Curvature on Surfaces: Theoretical Foundations for Discrete Curvature on Triangle Meshes

Introduction

Curvature represents a fundamental geometric property of surfaces, effectively describing local bending, feature formation, and shape variation. In smooth differential geometry, it is defined via derivatives of both the surface embedding and the normal field, relying on differentiability—a characteristic absent from triangle meshes. Nonetheless, curvature estimation on triangle meshes is essential in geometry processing, computer graphics, and finite element analysis, supporting applications such as segmentation, feature detection, smoothing, remeshing, and simulation. To achieve robust computation, a conceptual transition is required:

smooth curvature → curvature tensors → discrete curvature on meshes

This article establishes that transition, beginning with curvature of curves, extending to surfaces, introducing the shape operator, and examining challenges inherent to mesh-based curvature. This theoretical framework serves as preparation for the subsequent companion article, which details the construction of a practical curvature module for triangle meshes.

Curvature of Curves: The Starting Point

Curvature originates in the analysis of smooth curves.

$$\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$$

Consider a curve parameterized by arc length s.

The **unit tangent vector** is defined:

$$T(s) = \gamma'(s)$$

and the **curvature** corresponds to the magnitude of the derivative of this tangent vector:

$$\kappa(s) = \|T'(s)\|$$

The **principal normal** is

$$\mathrm{N}(s) = \frac{T'(s)}{\|T'(s)\|}$$

Curvature quantifies the rate at which the tangent direction changes (see Figure 1). For example, straight lines have zero curvature $k = 0$, while circles of radius r exhibit constant curvature $k = 1/r$.

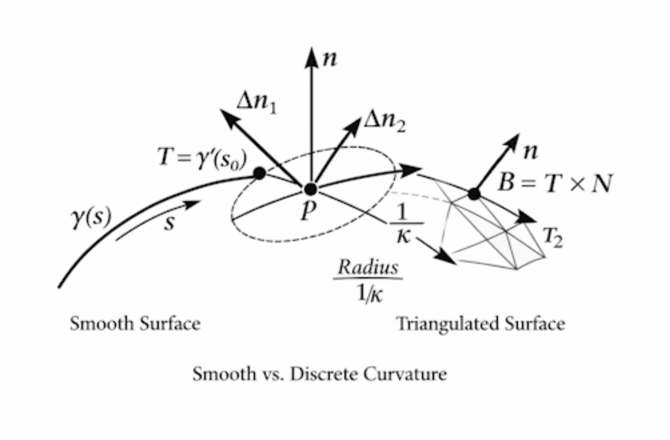

Figure 1: Tangent T, normal N, binormal B, and osculating circle of radius $1\ /\kappa$ at a point $p = \gamma(s_0)$ on a smooth curve

Curvature on Smooth Surfaces

Let S denote a smooth surface embedded in $\mathbb{R}^3$, with a point $\mathrm{p}\ \in S$. Infinitely many smooth curves can pass through p, each possessing individual curvature; however, only the component **aligned with the surface normal** encapsulates the surface's curvature.

Normal curvature

Given a unit tangent direction at p, and a corresponding curve γ on the surface such that $\gamma(p) = 0$ and $\gamma'(0) = v$, the **normal curvature** in direction is:

$$\kappa_n(v) = \langle \gamma''(0), n(p) \rangle,$$

where $n(p)$ is the unit normal of the surface at p.

Among all possible directions:

- The maximal normal curvature is the **first principal curvature** k_1.
- The minimal value is the **second principal curvature** k_2.

These extremal values occur in mutually orthogonal **principal directions**, forming the principal curvature frame (see Figure 2).

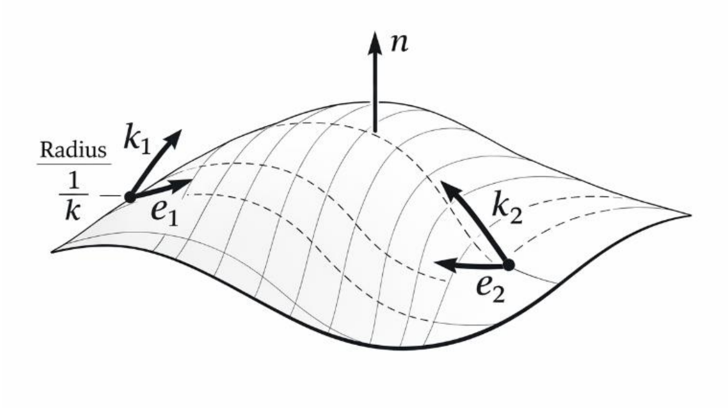

Figure 2: Principal curvature directions e_1,e_2 and corresponding curvatures k_1,k_2 on a smooth surface patch

Mean and Gaussian curvature

Two scalar invariants summarize the local geometry:

$$\mathrm{H} = \frac{k_1 + k_2}{2}, \qquad K = k_1 k_2.$$

These invariants assist in classifying local surface shape.

Shape type	Gaussian curvature (K)	Description
Elliptic	$\mathrm{K} > 0$	Dome-like, both curvatures same sign
Hyperbolic	$\mathrm{K} < 0$	Saddle-like
Parabolic	$\mathrm{K} = 0$	Cylindrical
Planar	$k_1 = k_2 = 0$	Flat

The Shape Operator: The Core Object

The most sophisticated formulation of curvature arises through the **shape operator** (Weingarten map).

Definition

Let $\mathrm{n}: \mathrm{S} \to \mathbb{S}^2$ be the unit normal field. The shape operator is:

$$S_p: T_p\mathrm{S} \to T_p\mathrm{S}, \qquad S_p(v) = -D_v\mathrm{n},$$

the negative directional derivative of the normal field.

Significance

The eigenvalues of S_p are the principal curvatures $\mathrm{k}_1,\ \mathrm{k}_2$, and the eigenvectors identify their directions.

The **second fundamental form** is

$$\mathrm{II}(u, v) = \langle S(u), v \rangle.$$

Thus:

Curvature is fundamentally encoded in variations of the normal field.

The shape operator succinctly captures these variations (see Figure 3), forming the basis of modern discrete curvature estimators.

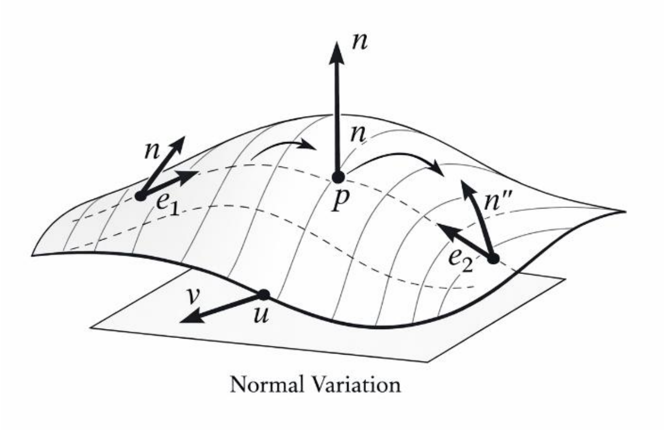

Figure 3: Variation of the surface normal around a point , illustrating the geometric meaning of the shape operator $S = -Dn$

Challenges of Curvature Estimation on Triangle Meshes

Triangle meshes are inherently **piecewise linear** structures. They lack:

- smooth derivatives,
- continuous tangent planes,
- consistent normals across edges,
- intrinsic curvature.

Despite these limitations, there is a persistent need to approximate smooth curvature using discrete representations.

Key Challenges

- **Normal discontinuity:** Normals exhibit abrupt changes along edges.
- **Neighborhood ambiguity:** “Locality” is ill-defined on irregular meshes.
- **Noise sensitivity:** Curvature magnifies noise artifacts.
- **Sampling irregularity:** Variability in triangle size and shape affects accuracy.
- **Absence of smoothness:** The shape operator requires derivatives unavailable on piecewise linear surfaces.

Discontinuous normals at mesh edges (see Figure 4) render accurate curvature estimation particularly challenging.

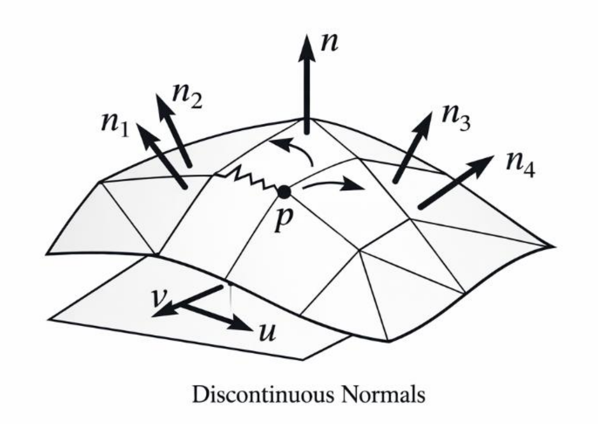

Figure 4: Face normals on a triangle mesh showing discontinuities across edges.

Discrete Curvature: Methods and Concepts

Several methodologies exist for approximating curvature on triangle meshes:

1. Angle deficit (Gaussian curvature)

$$\mathrm{K}(v) \approx \frac{2\pi - \sum \theta_i}{A_v}$$

Provides Gaussian curvature exclusively at vertices, with limited scope.

2. Cotangent Laplacian (mean curvature normal)

$$\mathrm{Hn} \approx \frac{1}{2A_v} \sum_{(i,j)} \left(\cot \alpha_{ij} + \cot \beta_{ij}\right)\left(x_i - x_j\right)$$

A widely adopted discrete differential operator.

3. Quadric fitting

Employs local polynomial surface fitting and analytical extraction of curvature.

4. PCA-based methods

Estimates curvature directions via covariance analysis of points or normals.

5. Normal cycle curvature tensor

Integrates normal variation throughout the neighborhood, closely approximating the shape operator.

Among these, the normal cycle approach aligns most closely with the principles of smooth differential geometry and forms the foundation of the forthcoming curvature module.

The robustness and theoretical grounding of these approaches vary considerably (see Figure 5).

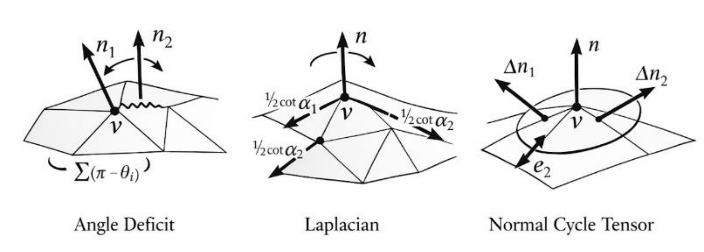

Figure 5: Three families of discrete curvature estimators: angle deficit, cotangent Laplacian, and normal cycle tensor.

The Discrete Curvature Tensor: Mathematical Formulation

In the context of smooth surfaces, curvature is encapsulated within the **shape operator**,

$$\mathrm{S} = -\mathrm{D}\,\mathrm{n},$$

representing the negative derivative of the normal field. On triangle meshes, the normal field is piecewise constant with its derivative distributed along edges. Approximating the smooth operator entails integrating normal variation over localized neighborhoods to construct a **discrete curvature tensor**.

Multiple formulations exist within the literature. Below, we present the most prevalent and theoretically robust variants.

1. Normal Variation Across Edges

For a vertex v, let incident edges e_i be shared by faces with normals n_i^+ and n_i^-. The **normal jump** across edge e_i is:

$$\Delta n_i = n_i^+ - n_i^- .$$

Let t_i denote the **unit tangent direction** along edge e_i and l_i the edge length.

The underlying principle is:

Normal variation across an edge approximates the directional derivative of the normal field.

Therefore, a discrete approximation to the shape operator's contribution from edge e_i is:

$$S_i \approx \frac{\Delta n_i \otimes t_i}{\mathrm{l}_i}$$

where $\otimes$ represents the outer product.

Each edge provides a directional term capturing normal variation (see Figure 6).

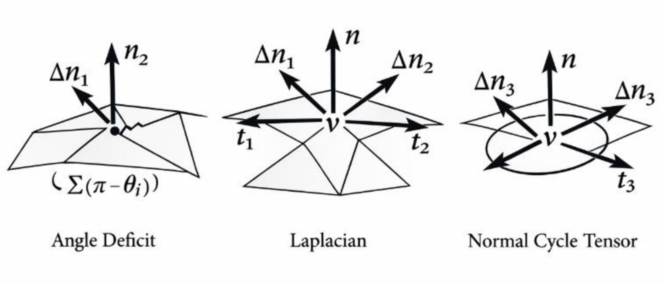

Figure 6:Edge-based approximation of the shape operator using normal jumps.

2. Area-Weighted Curvature Tensor

To aggregate contributions from all edges surrounding vertex v, weights based on the **dual area** A_v (often the mixed *Voronoi* area) are applied:

$$A_v = \frac{1}{8}\sum_{(i,j)}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right)\left\|x_i - x_j\right\|^2 .$$

The **discrete curvature tensor** thus becomes:

$$S_v = \frac{1}{2A_v}\sum_{e_i \in 1\text{-ring}(v)} \ell_i\, , (\Delta n_i \otimes t_i)$$

This is the discrete analogue of the smooth shape operator.

Interpretation

- t_i delineates movement direction along the surface.
- Δn_i describes normal rotation.

The tensor S_v characterizes how normal rotation depends on directionality.

3. Symmetrized Curvature Tensor

To retain symmetry—an intrinsic property of the continuous shape operator—the discrete tensor is often symmetrized:

$$\widetilde{S_v} = \frac{1}{2}(S_v + S_v^{\top})$$

This adjustment enhances numerical stability and guarantees real eigenvalues.

4. Principal Curvatures and Directions

Eigenanalysis of $\widetilde{S_v}$ yields the discrete principal curvatures and directions:

$$\widetilde{S_v}, e_1 = k_1 e_1, \qquad \widetilde{S_v}, e_2 = k_2 e_2.$$

Scalar curvatures are easily extracted:

$$\mathrm{H} = \frac{k_1 + k_2}{2}, \qquad K = k_1 k_2.$$

Visualizations may depict this tensor as an ellipse in the tangent plane (see Figure 7).

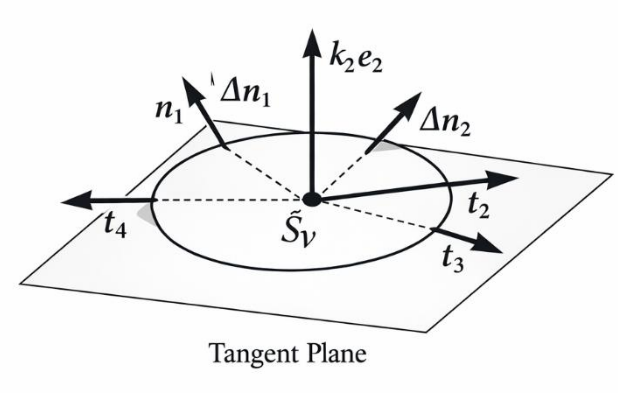

Figure 7: *Symmetrized curvature tensor* $\widetilde{S_v}$ *visualized as an ellipse in the tangent plane, with principal directions* e_1, e_2 *and magnitudes* k_1, k_2.

5. Tensor from Normal Covariance (Normal Cycles)

An alternative formulation uses the **covariance of normals** in the 1-ring:

$$C_v = \sum_{f \in 1\text{-ring}(v)} A_f \,, (n_f - \bar{n}) \otimes (n_f - \bar{n}),$$

where

$$\bar{n} = \frac{1}{\sum A_f} \sum A_f n_f.$$

This tensor records normal variability around the vertex. Projection onto the tangent plane yields a curvature tensor:

$$S_v = P_v, C_v, P_v$$

where

$$P_v = \mathrm{I} - n_v \otimes n_v$$

denotes tangent-plane projection.

This robust formulation is extensively utilized in geometry processing and accurately reflects curvature information (see Figure 8).

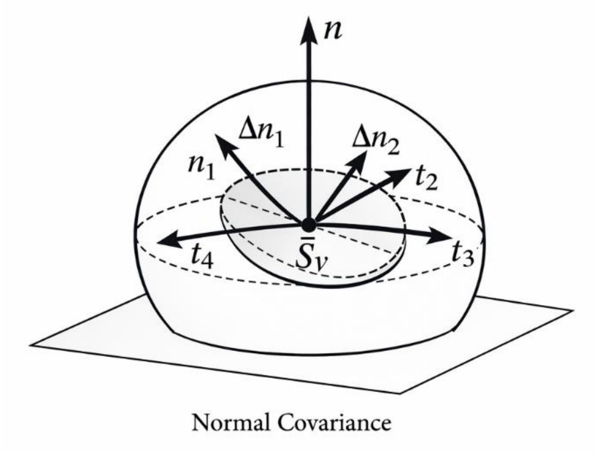

Figure 8: *Distribution of face normals on the unit sphere and their covariance ellipse.*

6. Tensor from Edge-Based Normal Cycles

A refined approach integrates normal variation along edges:

$$C_v = \sum_{e_i} \theta_i \, \ell_i (b_i \otimes b_i),$$

where:

- θ_i is the dihedral angle at edge e_i,
- b_i is the unit binormal direction,
- ℓ_i denotes edge length.

Then:

$$S_v = \frac{1}{A_v} P_v C_v P_v.$$

This is the discrete counterpart to smooth integration of the second fundamental form over neighborhoods, reliably mirroring smooth curvature behavior (see Figure 9).

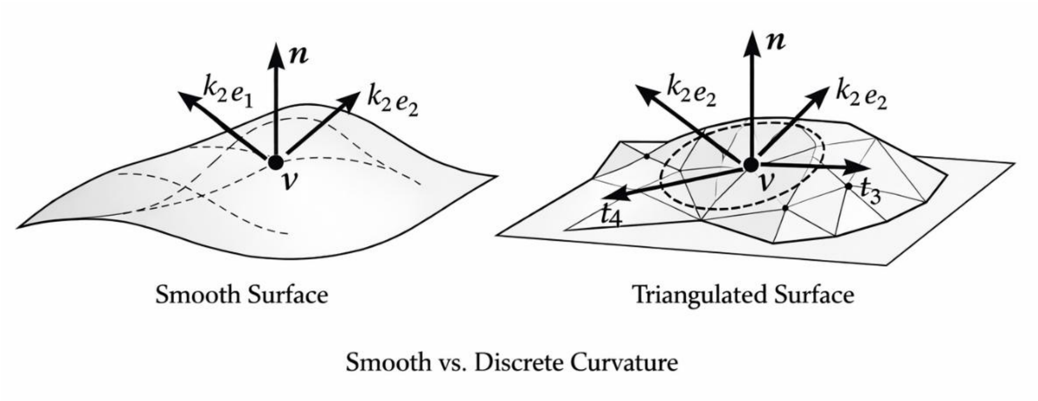

Figure 9: Comparison of curvature directions on a smooth surface and a triangulated approximation

Conclusion

Curvature constitutes a profound geometric concept grounded in differential geometry. A comprehensive understanding of curvature on smooth surfaces—including normal curvature, principal curvatures, and the shape operator—provides the necessary conceptual foundations for developing robust discrete curvature estimators for triangle meshes.

Forthcoming Developments

The next article will address the implementation of a practical curvature module designed to:

- construct discrete curvature tensors,
- extract principal curvatures and directions,
- manage noise and sampling irregularities,
- integrate seamlessly into finite element and segmentation pipelines.